THE UNIVERSITY OF ALBERTA

RESULTS CONNECTED WITH LINEAR DIFFERENCE EQUATIONS

by

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The undersigned certify that they have read and recommend to the Faculty of Graduate Studies for acceptance a thesis entitled "Results Connected With Linear Difference Equations", submitted by David A. Klarner, B.A., in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

This thesis falls naturally into two parts: in the first two chapters we treat certain enumeration problems in combinatorial geometry which involve the theory of linear difference equations with constant coefficients; in the last two chapters we study certain questions treating the formal side of this theory. For example, we show that the class C of all sequences which satisfy linear difference equations with constant coefficients form a ring under term by term addition and multiplication; i.e., if $\{a_n\}$ and $\{b_n\}$ are in C, then $\{a_n + b_n\}$ and $\{a_n \cdot b_n\}$ are also in C. Thus, in Chapter III we are led to seek the linear difference equations satisfied by powers of elements in C. In Chapter IV we make use of the results of Chapter III to find the value of a certain type of determinant which has appeared in various special forms in the recent literature.

ACKNOWLEDGEMENTS

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INTRODUCTION

The main result of Chapter I is an improved lower bound for $p(n)$, the number of distinct ways one can join n squares edge on edge in the plane. We show that $p(n) > (3.2)^n$ for sufficiently large n; it is already known that $p(n) < 4^n$ and it is conjectured that the lower bound is considerably greater than the one we provide. In the last part of this chapter we list some open questions of enumerative combinatorial geometry.

In Chapter II we develop a method for determining the number of ways one can cover a rectangle with a set of congruent figures. Golomb found the number of ways one could cover a $2 \times k$ rectangle with 1×2 rectangles; we go further than this and employ our method to find the number of ways of covering a $m \times \ell$ rectangle with $1 \times k$ rectangles for all values of m between 1 and 2k.

In Chapter III, we shift our attention to formal questions dealing with the theory of linear difference equations with constant coefficients (we will shorten this last phrase to "linear difference equations" and mean the same thing). The main results deal with the linear difference equation satisfied by the k^{th} powers of elements of a sequence which satisfies a linear difference equation; this is done by solving the equivalent problem of finding the generator of such a sequence (the problems are equivalent in the sense that one can easily deduce the relevant difference equation from the generating function once it is known).

In Chapter IV we make use of the results in Chapter III to find the value of certain k^{th} order determinants having elements taken from sequences which satisfy linear difference equations of the k^{th} order. Special cases of our results have appeared in the recent literature.

CHAPTER I

SOME RESULTS CONCERNING POLYOMINOES

Introductory Remarks: An n-omino is a plane figure composed of n connected unit squares joined edge on edge. In the early nineteen hundreds, Henry Dudeney, the famous British puzzle expert, and the Fairy Chess Review popularized problems involving n-ominoes which they represented as figures cut from checkerboards. Solomon Golomb seems to have been the first mathematician to treat the subject seriously when as a graduate student at Harvard in 1954, he published "Checkerboards and Polyominoes" in the American Mathematical Monthly. Since 1954, several articles have appeared, (see bibliography); in particular, R.C. Read [13] and Murray Eden [4] have discussed the problem of finding or estimating the number $p(n)$ of n-ominoes for a given n. From their results it is now known that for large n

$$c_1^n < p(n) < c_2^n$$

where c_1 and c_2 are certain positive constants greater than 1. In the first part of this chapter we enumerate a subset of n-ominoes and provide an improved lower bound for $p(n)$; later we discuss other problems of this sort and conclude with a brief exposition of problems dealing with configurations of n-ominoes.

Moser's Board Pile Problem: In the following it will be convenient to have certain conventions. We say the region between $y = n-1$ and $y = n$ is the n^{th} row and call a rectangle of width one a strip. The first square on the left in a strip located in a row is

called the initial square of the strip; an n-omino is located in the plane when some square in the n-omino exactly covers a square in the plane lattice. The set of all incongruent n-ominoes will be denoted by $P(n)$ and for convenience we think of the elements of $P(n)$ located in arbitrary regions of the plane. Ignoring changes in position due to translations, each element of $P(n)$ has eight or less positions with respect to 90° rotations about the origin and reflections along the x or y axes; taking two n-ominoes to be distinct if one cannot be translated to cover the other, we find a new set $S(n)$ from $P(n)$ by including rotations and reflections of n-ominoes in $P(n)$ in $S(n)$.

The problem which is now to be discussed was probably first posed by Leo Moser in private correspondence with the present author; later he posed it in a different form at the 1963 Number Theory Conference held at the University of Colorado. Eden [4] also discusses the problem, but his results are not as complete as those given here. The problem is to enumerate a subset $B(n)$ of $S(n)$ which contains n-ominoes having the property that they can be translated in such a way that they are entirely in the first and second quadrants with exactly one strip in the first row with its initial square at the origin and each row after the first has no more than one strip in it. Such n-ominoes may be visualized as side elevations of board piles consisting of boards of various lengths which generally have not been stacked carefully.

Moser noted that if $b(n)$ denotes the number of elements in $B(n)$, then

$$(1) \qquad b(n) = \sum (a_1 + a_2 - 1)(a_2 + a_3 - 1) \ldots (a_{i-1} + a_i - 1)$$

where the summation extends over all compositions $a_1 + a_2 \ldots + a_i = n$ of n. The relation in (1) can be established by the following combinatorial argument. For each composition $a_1 + a_2 + \quad + a_i$ of n there is a subset of B(n) consisting of n-ominoes which have a strip of a_t squares in the t^{th} row (t = 1, 2, ..., i); the number of n-ominoes in each of these subsets is 1 if i = 1 which corresponds to the value of the empty product in the sum (in this there is a strip n units long in the first row) and $(a_1 + a_2 - 1)(a_2 + a_3 - 1) \ldots (a_{i-1} + a_i - 1)$ if $i \geq 2$. This follows since there are exactly $(a_{t-1} + a_t - 1)$ ways to join the strip of a_t squares in the t^{th} row to the strip of a_{t-1} squares in the row below and the total number of ways to connect up the strips to form an n-omino would be the product of all of these alternatives. The subsets corresponding to the compositions of n are exhaustive and disjoint in B(n), so that b(n) is the sum of the number of elements in each subset, which is (1).

The relation for b(n) given by (1) does not furnish a very handy device for computing b(n), but as Eden has shown it is helpful in estimating b(n). Rather than attempt to sum (1) by purely algebraic manipulations, we retain the geometric interpretation of the problem so that combinatorial arguments can be more easily applied toward finding a recursion relation for b(n).

To find a recursion relation for b(n) we define subsets $B_r(n)$ (r = 1, 2, ... , n) of B(n) which contain n-ominoes with a strip of exactly r squares in the first row and let $b_r(n)$ denote the

number of elements in $B_r(n)$. It is obvious that the subsets $B_r(n)$ $(r = 1, 2, \ldots, n)$ are exhaustive and disjoint in $B(n)$ so that we have immediately

$$b(n) = \sum_{r=1}^{n} b_r(n) . \tag{2}$$

By definition of $B_n(n)$, $b_n(n) = 1$. Consider the elements of $B_r(n)$ with $r < n$; each element of $B_r(n)$ consists of a strip of r squares in the first row with some element of $B(n-r)$ located in the rows above the first. The situation can be appraised more concisely when one considers the number of ways an element of the subset $B_i(n-r)$ of $B(n-r)$ can be attached to the strip of r squares in the first row so that the n-ominoes formed will be an element of $B_r(n)$. Clearly this can be done in $r + i - 1$ ways, so that exactly $(r + i - 1)\, b_i(n-r)$ of the elements of $B_r(n)$ have an element of $B_i(n-r)$ connected to the strip of r squares in the first row. Since the subsets $B_i(n-r)$ $(i = 1, 2, \ldots, n-r)$ of $B(n-r)$ are exhaustive, disjoint subsets, it follows that

$$b_r(n) = \sum_{i=1}^{n-r} (r + i - 1)\, b_i(n-r) \qquad \text{for } r < n . \tag{3}$$

It will be seen presently that the relations in (2) and (3) are enough to find the desired recursion relation for $b(n)$. Before this result can be given, we have to prove a few lemmas.

<u>Lemma 1</u>: If $n > 1$, $b_r(n) - b_{r-1}(n-1) = b(n-r)$.

<u>Proof</u>: Using (3) it is seen that

$$b_r(n) - b_{r-1}(n-1) = \sum_{i=1}^{n-r} (r+i-1)b_i(n-r) - \sum_{i=1}^{n-r} (r+i-2)b_i(n-r)$$

$$= \sum_{i=1}^{n-r} b_i(n-r) \; ;$$

but according to (2), the last expression is precisely $b(n-r)$, so the proof is finished.

<u>Lemma 2</u>: If $n > 1$, $b(n) = 2\,b(n-1) + b_1(n) - b_1(n-1)$.

<u>Proof</u>: Using relations for $b(n)$ and $b(n-1)$ given by (2), it is seen that

$$(5) \qquad b(n) - b(n-1) = \sum_{i=1}^{n} b_i(n) - \sum_{i=1}^{n-1} b_i(n-1)$$

$$= b_1(n) + \sum_{i=2}^{n-1} \left\{ b_i(n) - b_{i-1}(n-1) \right\} \; ;$$

but according to Lemma 1, $b(n-i)$ can be substituted for $b_i(n) - b_{i-1}(n-1)$ in the last member of (5) so that making this substitution and transposing $-b(n-1)$ from the first to the last member gives

$$(6) \qquad b(n) = b_1(n) + \sum_{i=1}^{n-1} b(n-i).$$

Now using relations given by (6) for $b(n)$ and $b(n-1)$ we have

$$(7) \qquad b(n) - b(n-1) = b_1(n) + \sum_{i=1}^{n-1} b(n-i) - b_1(n-1) - \sum_{i=1}^{n-2} b(n-1-i)$$

$$= b_1(n) - b_1(n-1) + b(n-1) \; ;$$

the desired result is obtained by adding $b(n-1)$ to the first and last members of (7).

<u>Lemma 3</u>: $\quad b_1(n) = 4\, b_1(n-1) - 4\, b_1(n-2) + b_1(n-3) + 2\, b(n-3)$.

<u>Proof</u>: Taking $r = 1$ in (3) gives an expression for $b_1(n)$; namely,

$$b_1(n) = \sum_{i=1}^{n-1} i\, b_i(n-1) \; . \tag{8}$$

Using relations for $b_1(n)$ and $b_1(n-1)$ given by (8) and substituting $b(n-2-i)$ for $b_{i+1}(n-1) - b_i(n-2)$ and $b(n-1)$ for $\sum_{i=1}^{n-1} b_i(n-1)$ when they occur, it is seen that

$$\begin{aligned} b_1(n) - b_1(n-1) &= \sum_{i=1}^{n-1} i\, b_i(n-1) \quad \sum_{i=1}^{n-2} i\, b_i(n-2) \\ &= \sum_{i=1}^{n-1} b_i(n-1) + \sum_{i=1}^{n-2} i\, b_{i+1}(n-1) - \sum_{i=1}^{n-2} i\, b_i(n-2) \\ &= b(n-1) + \sum_{i=1}^{n-2} i \left\{ b_{i+1}(n-1) - b_i(n-2) \right\} \\ &= b(n-1) + \sum_{i=1}^{n-2} i\, b(n-2-i) \; . \end{aligned} \tag{9}$$

Adding $b_1(n-1)$ to each member of the equality and dropping the last term in the sum in the right member of (9) (since $b(0) = 0$) a new relation for $b_1(n)$ is obtained,

$$b_1(n) = b_1(n-1) + b(n-1) + \sum_{i=1}^{n-3} (n-2-i)\, b(i) \; . \tag{10}$$

This time using expressions for $b_1(n)$ and $b_1(n-1)$ given by (10) and again writing a relation for $b_1(n) - b_1(n-1)$, one obtains after a few algebraic manipulations

$$(11) \qquad b_1(n) = 2b_1(n-1) - b_1(n-2) - 2b(n-2) + \sum_{i=1}^{n-1} b(i).$$

Repeating the same procedure as before only this time using expressions for $b_1(n)$ and $b_1(n-1)$ given by (11) yields

$$(12) \quad b_1(n) = 3b_1(n-1) - 3b_1(n-2) + b_1(n-3) + b(n-1) - 2b(n-2) + 2b(n-3);$$

but by Lemma 2, $b(n-1) - 2b(n-2) = b_1(n-1) - b_1(n-2)$ so that substituting the latter quantity for the former in (12) gives the desired result.

<u>Theorem 1</u>: $b(1) = 1$, $b(2) = 2$, $b(3) = 6$, $b(4) = 19$, and $b(n) = 5b(n-1) - 7b(n-2) + 4b(n-3)$ for $n > 4$.

<u>Proof</u>: The values of $b(i)$ $(i = 1, 2, 3, 4)$ can be computed directly from (1) or by taking $b(1) = b_1(1) = 1$ the relations in (2) and (3) can be used together for the same purpose. Lemmas 2 and 3 provide the linear difference equations involving $b_1(n)$ and $b(n)$ which can be used to find

$$(13) \qquad b(n) = 5b(n-1) - 7b(n-2) + 4b(n-3),$$

$$(14) \qquad b_1(n) = 6b_1(n-1) - 12b_1(n-2) + 11b(n-3) - 4b(n-4),$$

which completes the proof.

The auxilliary equation for (13) has one real root greater than 3.2 so that for n sufficiently large

(15) $$b(n) > (3.2)^n .$$

We conclude from earlier remarks that B(n) contains at least b(n)/8 incongruent n-ominoes, so that we can also replace b(n) in (15) with p(n).

Having disposed of the more difficult problem first, we now turn attention to solving an easier and related problem which was posed and solved by Moser.

Let C(n) be the subset of B(n) which contains all n-ominoes having the property that the initial square of the strip in the k^{th} row is no further to the left that the initial square of the strip in the $(k-1)^{st}$ row. Recall from the definition of B(n) that the initial square of the strip in the first row is always located at the origin. Using a combinatorial argument similar to the one provided for the proof of (1), it is easy to prove

(16) $$c(n) = \sum_{a_1+a_2+\cdots+a_i=n} a_1 a_2 \cdots a_{i-1} ,$$

where c(n) denotes the number of elements in C(n). Applying the methods he gave in [8], Moser was able to show from (16):

<u>Theorem 2:</u> c(n) is equal to the $(2n-1)^{st}$ Fibonacci number.

We will give an alternate proof using the same idea used in the proof of Theorem 1. Let $C_i(n)$ be the subset of C(n) which contains all n-ominoes having strips of exactly i squares in the first row. Clearly the subsets $C_i(n)$ $(i = 1, 2, \ldots , n)$ are exhaustive and disjoint in C(n) so that letting $c_i(n)$ denote the number of elements in

$C_i(n)$ we have

$$(17) \qquad c(n) = \sum_{i=1}^{n} c_i(n) .$$

Next, it is easy to see that $c_n(n) = 1$, and for $i < n$, $c_i(n) = i\, c(n-i)$ since each element of $C(n-i)$ can be joined exactly i ways to the strip of i squares in the first row so as to form an element of $C_i(n)$; the n-ominoes thus formed obviously comprise all the elements of $C_i(n)$. Substituting the expressions just found for $c_i(n)$ into (17) we obtain

$$(18) \qquad c(n) = 1 + \sum_{i=1}^{n-1} i\, c(n-i) .$$

Using expressions for $c(n)$ and $c(n-1)$ given by (18) we can combine the sums in $c(n) - c(n-1)$ to find

$$c(n) - c(n-1) = \sum_{i=1}^{n-1} c(i),$$

or

$$c(n) = c(n-1) + \sum_{i=1}^{n-1} c(i) .$$

Now using expressions for $c(n)$ and $c(n-1)$ given by (19) we can combine the sums in $c(n) - c(n-1)$ and deduce

$$(20) \qquad c(n) = 3\, c(n-1) - c(n-2).$$

It is easy to prove that the Fibonacci numbers with odd indices satisfy the recurrence relation in (20). Also, using (16) we find $c(1) = f_1$ and $c(2) = f_3$ (f_i denotes the i^{th} Fibonacci number as usual) so that the sequences $\{c_i\}$ and $\{f_{2i-1}\}$ must be identical.

N-ominoes Enclosed in Rectangles

R. C. Read [13] has treated the problem of enumerating the n-ominoes which "fit" into a p x q rectangle. An n-omino is said to <u>fit</u> in a p x q rectangle if it is the smallest rectangle in which the n-omino can be drawn with the sides of its squares parallel to the sides of the rectangle. Read's methods give exact counts of the n-ominoes in the sets considered; however, it is possible to obtain lower bounds for these numbers with less effort using similar ideas To illustrate we will consider the problem of estimating from below the number $s_2(n)$ of n-ominoes which fit in a 2 x k rectangle; we call this set of n-ominoes $S_2(n)$. Two elements are distinct if they are incongruent, so $S_2(n)$ is a subset of $P(n)$.

First, we observe that each element of $S_2(n)$ can be located entirely in the first quadrant in rows 1 and 2 with a square located at the origin. If each element of $S_2(n)$ is situated in the way just described in every way possible, a new set $U(n)$ is obtained where two elements are distinct if one does not exactly cover the other. Clearly, $u(n)$, the number of elements in $U(n)$, is less than or equal to $4s_2(n)$. Now $U(n)$ can be divided into two sets $U''(n)$ and $U'(n)$ consisting respectively of n-ominoes having and not having a square in the second row attached to the square at the origin. Let the number of elements in $U'(n)$ and $U''(n)$ be $u'(n)$ and $u''(n)$ respectively Now it is easy to see that

$$(21) \qquad u'(n) = u'(n-1) + u''(n-1)$$

since every element of $U''(n-1)$ and $U'(n-1)$ can be translated a unit

to the right of the origin and a square located at the origin to give an element of $U'(n)$ and every element is obviously obtained in this fashion. It is also easy to prove

(22) $$u''(n) = 2u'(n-2) + u''(n-2)$$

since every element of $U''(n-2)$ and every element of $U'(n-2)$ and its horizontal reflection can be translated a unit to the right of the origin and two squares added (one at the origin, the other attached above it) to form every element of $U''(n)$.

Using (21) and (22) we can find

(23) $$u'(n) = u'(n-1) + u'(n-2) + u'(n-3)$$

and

(24) $$u''(n) = u''(n-1) + u''(n-2) + u''(n-3),$$

so that it becomes evident from $u(n) = u'(n) + u''(n)$ that

(25) $$u(n) = u(n-1) + u(n-2) + u(n-3).$$

Since $u(n)/4 \leq s_2(n)$, (25) provides a relation for estimating $s_2(n)$. The same procedure can be used for estimating the number of elements in $S_k(n)$ consisting of n-ominoes which fit in k x q rectangles.

N-omino Configurations: Problems involving n-omino configurations have enjoyed a great popularity among mathematical recreationists [6], [8]. We plan to devote a small amount of space to giving an exposition of problems which may be of interest to the

mathematician. Generally these problems have the following form: given a region of area A and a set of n-ominoes having a combined area also A; can one cover the region with the set?

We say a set exactly covers a region when there is no overlap and no part of the region is left uncovered. It would be interesting to know necessary conditions that an n-omino be such that an unlimited number of copies could be used to exactly cover the plane. A related problem is to determine necessary conditions that some number of copies of a given n-omino could be used to exactly cover a rectangle. Thus, some easily proved necessary conditions are given by:

(i) if an n-omino has two lines of symmetry and a set of these n-ominoes exactly covers a rectangle, then the n-omino is itself a rectangle.

(ii) if an n-omino fits in a $p \times q$ rectangle and covers diagonally opposite corners of the rectangle, and a set of these n-ominoes can be used to exactly cover a rectangle, then the n-omino is itself a rectangle.

A rectangle exactly covered with a set of congruent n-ominoes is minimal when no rectangle of smaller area can be exactly covered with a set of the same n-ominoes containing fewer elements. It is easy to prove that there is an unlimited number of minimal rectangles involving either two or four n-ominoes. Figures 1, 2, 3 and 4 show instances of minimal rectangles involving more than four n-ominoes. Are there infinitely many cases of minimal rectangles

which involve more than four n-ominoes (no two cases involving similar n-ominoes)? Are there minimal rectangles involving an odd number of n-ominoes which are not themselves rectangles?

Note that the configurations depicted in figures 1, 2, 3 and 4 are symmetric with respect to the centers of the rectangles. Can this always be done in minimal rectangles?

Generalizations of n-ominoes: In [7], Golomb suggests that one could try to determine or estimate the number of distinct ways n equilateral triangles or n regular hexagons could be simply connected edge on edge. Using 1, 2, 3, 4, 5 or 6 hexagons 1, 1, 3, 7, 22 or 83 combinations respectively result; so far no upper or lower bounds for the terms of this sequence have been given.

There is no reason why regular k-gons could not be used for cells in such combinatorial problems; overlapping of cells could be permitted so long as no cell exactly covered another. Thus, where at most four squares or three hexagons might have a vertex in common, at most ten pentagons might have a vertex in common. The number of distinct ways to join two regular k-gons is one; the number of ways to join three regular k-gons is the greatest integer in $k/2$. Perhaps it would not be difficult to determine in how many ways four or five regular k-gons could be joined together edge on edge so that distinct simply connected figures are formed.

Still another generalization of n-ominoes which seems not to have been considered is joining squares together edge on edge in three or more dimensions. The number of ways of joining k cubes

face on face in three dimensions (including mirror images of some pieces) is 1, 1, 2, 8, 29, and 166 for $k = 1, 2, 3, 4, 5$, and 6 respectively; no bounds have been given for the terms of this sequence nor has much been done in a serious vein connected with the packing of space with these three dimensional analogues of polyominoes.

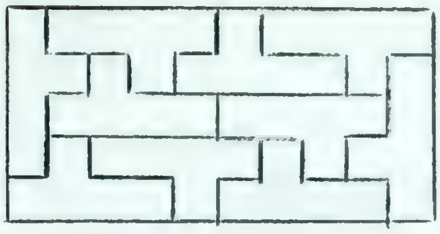


Figure 1

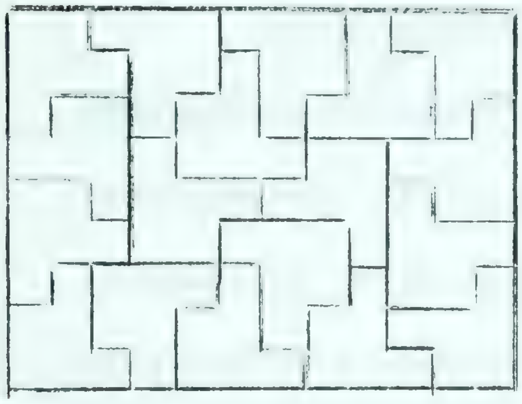


Figure 2

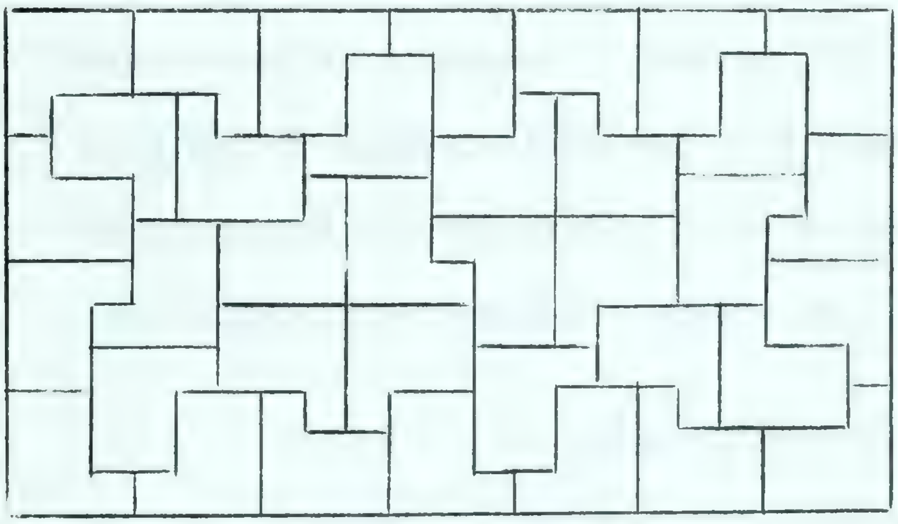


Figure 3

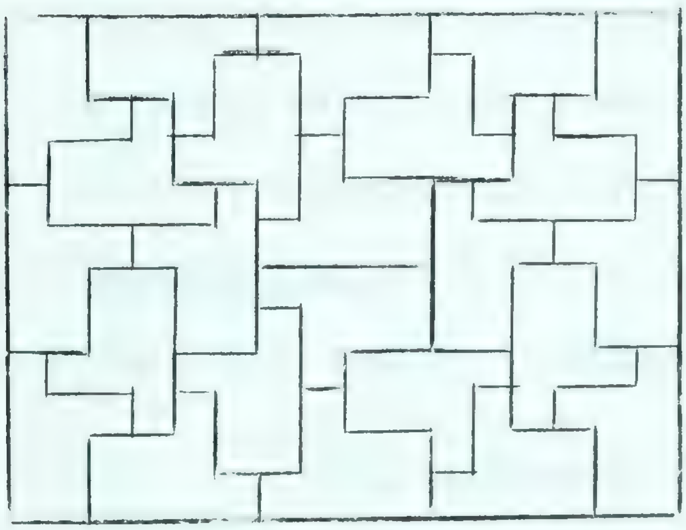


Figure 4

Figures 1 - 4 are minimal configurations involving pentominoes, hexominoes, heptominoes and octominoes respectively.

CHAPTER II

COVERING A RECTANGLE WITH A SET OF CONGRUENT FIGURES

Introductory Remarks: Throughout this chapter we will be concerned with geometric figures which can be formed by joining unit squares edge on edge in the plane. We will say that a rectangle R is an $m \times \ell$ f-configuration if R is located at the origin with a side of length ℓ extended along the x axis and a side of length m extended along the y axis and if R is exactly covered with a set of congruent f figures. (A figure R is exactly covered, covered, or packed with a set of f figures if every square of R is superimposed by some square in an f-figure and if every square in an f-figure superimposes a square in R.)

Two $m \times \ell$ f-configurations are distinct if at least one of the f-figures in one does not cover an f-figure in the other. If R is an $m \times \ell$ f-configuration and R can be cut into two rectangles R' and R" with a line L parallel to the y axis so that R' or R" is an $m \times \ell$ f-configuration with $0 < \ell' < \ell$, then L is called a fracture line and R is said to be composite. If R has no fracture lines, R is said to be primitive.

It is the purpose here to develop a method for finding the number of $m \times \ell$ f-configurations. This number will be denoted by W_ℓ throughout, although the meaning of W_ℓ will change from one case to another depending on m and the f figure under consideration.

The method is as follows: the fracture lines of a given $m \times \ell$ f-configuration F divide F into a sequence of primitive $m \times a_i$ f-configurations which have been translated an integral distance to the right of the origin along the x axis; this sequence is the <u>decomposition</u> of F. Every F has a decomposition corresponding to a composition $a_1 + a_2 + \ldots + a_i = \ell$; that is, the fracture lines of F are at $y = a_1$, $y = a_1 + a_2$, ... and the right boundary of F is at $y = a_1 + a_2 + \ldots + a_i = \ell$. Letting w_a denote the number of primitive $m \times a$ f-configurations, we see that the number of F's having decompositions corresponding to the composition $a_1 + a_2 + \ldots + a_i = \ell$ is $w_{a_1}, w_{a_2}, \ldots w_{a_i}$; hence, the number of $m \times \ell$ f-configurations is

$$(1) \qquad W_\ell = \sum_{a_1+a_2+\ldots+a_i=\ell} w_{a_1} \cdot w_{a_2} \cdot \ldots w_{a_i} ,$$

where the summation extends over the compositions of ℓ. The following theorem is sometimes useful in obtaining a generating function for $\{W_\ell\}$.

<u>Theorem:</u> If

$$(2) \qquad p(x) = \sum_{n=1}^{\infty} p_n x^n , \quad m(x) = \sum_{n=1}^{\infty} m_n x^n$$

then

$$(3) \qquad p(m(x)) = \sum_{n=1}^{\infty} \left\{ \sum_{a_1+a_2+\ldots+a_k=n} p_k \cdot m_{a_i} \cdot m_{a_2} \ldots m_{a_k} \right\} x^n .$$

Proof:

$$(4) \qquad p(m(x)) = \sum_{n=1}^{\infty} p_n [w(x)]^n$$

$$= \sum_{n=1}^{\infty} p_n \sum_{k=1}^{\infty} \sum_{a_1+a_2+\ldots+a_n=k} m_{a_1} \cdot m_{a_2} \cdots m_{a_n} x^k$$

$$= \sum_{k=1}^{\infty} \left\{ \sum_{n=1}^{\infty} \sum_{a_1+\ldots+a_n=k} p_n \cdot m_{a_1} \cdot m_{a_2} \cdots m_{a_n} \right\} x^k$$

$$= \sum_{k=1}^{\infty} \left\{ \sum_{a_1+a_2+\ldots+a_n=k} p_n \cdot m_{a_1} \cdot m_{a_2} \cdots m_{a_n} \right\} x^k ,$$

where the second summation in the last member is understood to extend over all compositions of k.

When $p(x) = x(1-x)^{-1}$, $p_n = 1$ for all n so that (3) becomes

$$(5) \qquad \frac{m(x)}{1-m(x)} = \sum_{n=1}^{\infty} \left\{ \sum_{a_1+a_2+\ldots+a_k=n} m_{a_1} \cdot m_{a_2} \cdots m_{a_k} \right\} x^n .$$

This special case of our theorem was given by Moser and Whitney [12], and they also gave several examples of applications which could be made. In what follows we will write w(x) for the generator of $\{w_n\}$ and note that substituting w(x) for m(x) in (5) gives W(x) the generator of $\{W_\ell\}$ since the coefficient of x^ℓ is W_ℓ.

<u>Packing Rectangles with 1 x k Rectangles</u>: It will be supposed throughout this section that the f-figure in the preceding section is specialized to mean a 1 x k rectangle with k > 1. The

notation introduced already will be interpreted as applying only to this figure. Thus, our problem now is to determine the number W_ℓ of $(m \times \ell) - (1 \times k)$ rectangle-configurations.

For $1 \leq m < k$, it is obvious $W_n = 1$ or 0 depending on whether k does or does not divide n. Proof can be given if we note that $w_k = 1$ and $w_i = 0$ for $i \neq k$; hence, $w(x) = x^k$ and thus by (5)

$$(6) \qquad W(x) = \frac{w(x)}{1 - w(x)} = \frac{x^k}{1-x^k} = \sum_{n=1}^{\infty} W_n x^n ,$$

which implies the result.

For $m = k$, the result is not as obvious as before; however, it is clear that $w_1 = 1$, $w_k = 1$ and $w_i = 0$ for $i \neq 1, k$. Hence, $w(x) = x + x^k$ so that

$$(7) \qquad W(x) = \frac{x + x^k}{1 - x - x^k} = \sum_{n=1}^{\infty} W_n x^n ,$$

which implies

$$(8) \qquad W_1 = W_2 = \dots = W_{k-1} = 1, \quad W_k = 2, \text{ and}$$

$$W_n = W_{n-1} + W_{n-k} \qquad \text{for } n > k.$$

When $k = 2$, W_k becomes the k^{th} term of the Fibonacci sequence $\{1, 2, 3, 5, 8, \dots\}$; a combinatorial proof of this result has also been given by Golomb [9].

For $m = k+1$, the problem becomes more difficult. We first note that k must divide the area $\ell(k + 1)$ of the rectangle so that ℓ is evidently a multiple of k. Thus, $w_i = 0$ whenever

$i \neq ak$. It is obvious that there are exactly three $(k + 1) \times k$ configurations so that $w_k = 3$; for w_{ak} with $a > 1$ we prove that $w_{ak} = \binom{a + k - 3}{k - 2}$.

A $1 \times k$ rectangle in a covering will be said to be a horizontal or vertical tile when the side of length k is parallel to the x or y axis respectively. Consideration of the ways in which it is possible to pack the end of a $(k + 1) \times ak$ rectangle at the origin makes it immediately clear that the packing of the entire rectangle can be accomplished only by packing a $1 \times ak$ strip along the x axis or along the side of the rectangle above and parallel to the x axis; since the configurations in the latter are all 180° rotations of configurations in the former we restrict our attention to these for the moment. Thus, w_{ak} is twice the number of ways of covering a $k \times ak$ rectangle above a $1 \times ak$ strip packed with horizontal tiles so that no fracture lines are formed at $y = k$, $y = 2k$, ... ; this can be done only if exactly k vertical tiles are used, and the positions of two of these must be fixed so that they pack the ends of the rectangle. The remaining $k - 2$ vertical tiles are then separated into a sets having p_i elements in the i^{th} set with $0 \leq p_i \leq k-2$; the p_i vertical tiles in the i^{th} set are located above the i^{th} horizontal tile and appear between sets of k horizontal tiles (except at the ends, where one of the boundaries is the edge of the vertical tile located there).

The reason that exactly k vertical tiles are involved is as follows. We wish to pack a $k \times ak$ rectangle so that no fracture lines appear at $y = k, 2k, \ldots, (a-1)k$; in order to do this we

must locate horizontal tiles so that they cover each of these a - 1 lines -- and this can only be done if the horizontal tiles appear in k x k square blocks. Thus, $(a-1)k^2$ of the area ak^2 is covered with horizontal tiles; the remaining area k^2 cannot be covered with horizontal tiles since there would then be fracture lines at $y = k, 2k, \ldots$ and so it follows that k vertical tiles must be used in the packing in the manner already described.

From the previous argument we can conclude that w_{ak} is twice the number of ordered a-tuples $(p_k, p_2, \ldots p_a)$ with $o \leq p_i \leq k-2$ such that $\sum p_i = k-2$, but this is the coefficient of x^{k-2} in the expansion of $(x^o + x^1 + \ldots + x^{k-2})^a$ which is $\binom{a+k-3}{k-2}$. Thus, since

$$x(x^o + x^1 + \ldots)^{k-1} = \frac{x}{(1-x)^{k-1}} = \sum_{a=1}^{\infty} \binom{a+k-3}{k-2} x^a, \tag{9}$$

we can obtain a generating function for $\{w_n\}$ from (9) by making a few minor adjustments:

$$\begin{aligned} w(x) &= x^k + \frac{2x^k}{(1-x^k)^{k-1}} \\ &= 3x^k + \sum_{a=2}^{\infty} 2\binom{a+k-3}{k-2} x^{ak} \\ &= \sum_{n=1}^{\infty} w_n x^n . \end{aligned} \tag{10}$$

Using (5) we obtain

$$(11)\qquad W(x) = \frac{x^k(1-x^k)^{k-1} + 2x^k}{(1-x^k)^k - 2x^k}$$

$$= \sum_{n=1}^{\infty} W_n x^n .$$

By cross multiplying and equating the coefficients of x^n we can find the first few values of w_n and a linear difference equation with constant coefficients satisfied by $\{w_n\}$ by using (11). An explicit expression for w_n can be found in the usual way.

When $k = 2$, (11) gives a means for determining W_n the number of ways in which a $3 \times n$ rectangle can be packed with dominoes. In this case

$$(12)\qquad W(x) = \frac{3x^2 - x^4}{1 - 4x^2 + x^4} = \sum_{n=1}^{\infty} W_n x^n ,$$

which implies

$$(13)\qquad W_{2k-1} = 0, \quad W_2 = 3, \; W_4 = 11, \text{ and}$$

$$W_{2n} = 4\, W_{2n+2} - W_{2n} \qquad n = 1, 2, \ldots$$

Also, we have

$$(14)\qquad W_{2n} = \left\{ (3\theta_1 - 1)\theta_1^{n-1} - (3\theta_2 - 1)\theta_2^{n-1} \right\} / (\theta_1 - \theta_2$$

where θ_1 and θ_2 are the zeros of $y^2 - 4y + 1 = 0$.

When $k = 3$, (11) can be used to ascertain the number of ways a $4 \times 3a$ rectangle can be packed with rectangular trominoes; the sequence has in part the form

(15) $\{3, 13, 57, 247, 1077, 4701, \ldots\}$,

the elements of which satisfy the linear difference equation

(16) $$a_{n+3} = 5a_{n+2} - 3a_{n+1} + a_n .$$

When $m = k + b$ with $0 < b < k$, the situation is nearly the same as in the case just treated when $b = 1$. One can easily verify that a $(k + b) \times ak$ configuration must involve b $1 \times ak$ strips packed with horizontal tiles. This observation leads to

(17) $$w_k = b + 2, \quad w_{ak} = (b+1)\binom{a+k-3}{k-2} \quad \text{for } a > 1,$$

so that

(18) $$w(x) = x^k + \sum_{a=1}^{\infty} (b+1)\binom{a+k-3}{k-2} x^{ak}$$

$$= x^k + \frac{(b+1)x^k}{(1-x^k)^{k-1}}$$

$$= \sum_{n=1}^{\infty} w_n x^n .$$

Now we can obtain $W(x)$ from $w(x)$ in (18) as follows:

(19) $$W(x) = \frac{x^k(1-x^k)^{k-1} + (b+1)x^k}{(1-x^k)^k - (b+1)x^k}$$

$$= \sum_{n=1}^{\infty} W_n x^n .$$

If $b = 1$ in (19) we get (12). Taking $k = 4$, $b = 3$ in (19), W_n becomes the number of ways one can pack a $7 \times n$ rectangle with I-tetrominoes. We can deduce

$$(20) \qquad W_i = 0 \text{ if } i \neq 4t, \quad W_4 = 5, \quad W_8 = 37, \quad W_{12} = 269,$$

$$W_{16} = 1949, \text{ and for } n = 1, 2, \ldots$$

$$W_{4n+16} = 8\,W_{4n+12} - 6\,W_{4n+8} + 4\,W_{4n+4} - W_{4n}\,.$$

For $m = 2k$, the original problem becomes more difficult, but not impossible to solve by our methods. When $m = 2k + 1$, we have been unable to solve the problem even for $k = 2$. That is, how many distinct $5 \times \ell$ domino configurations are there ?

Covering Rectangles with Non-rectangular Polyominoes.

At this point, it should be clear that the method we are using is effective only as long as one can find w_n and a generation $w(x)$ for $\{w_n\}$; in general, this task is fairly simple when the width m of the $m \times \ell$ f-configuration is "small" compared to the size of f. This is usually the case since w_n displays "regular" behvaiour under these conditions. The problem becomes more difficult when the width of the configuration is "large", but even in these cases our method can sometimes be effectively applied toward estimating W_ℓ if upper and lower bounds for w_n can be found. In this concluding section we will indicate some results involving various non-rectangular polyominoes f; in every case the number of $m \times \ell$ f-configurations for m larger than that considered seems inaccessable by our methods.

In our first example we will consider m x ℓ L-tetrominoes configurations with m = 3; the cases when m = 1, 2 are trivial. (An L-tetromino is a set of 4 squares joined edge on edge in the plane which resembles an L in shape; see Figure 5.) Golomb [9] has shown that a necessary and sufficient condition that an a x b rectangle be exactly covered with L-tetrominoes is that 8 divide ab and a,b > 1. Thus, we see that $w_n = 0$ if $n \neq 8k$. It is easy to prove by a small number of trials that every 3 x 8k configuration is composed of a sequence of "links" which are shown in Figure 5. Each of these links contains a 2 x 4 rectangle which can be packed in either of two obvious ways. Once we have chosen the starting link (A or B in Figure 5) the choices of the remaining links are fixed except that the 2 x 4 rectangles in each link can be packed in two ways. From these remarks we can conclude that $w_{8k} = 2^{k+1}$ so that $W_{8k} = 4.6^{k-1}$.

Using L-trominoes to pack 4 x n rectangles, we have shown that the number of configurations W_n is given by

$$(21) \qquad W(x) = \frac{4x^3 - 14x^6}{1 - 8x^3 + 14x^6} = \sum_{n=1}^{\infty} W_n x^n ,$$

which implies $W_i = 0$ if $i \neq 3a$, $W_3 = 4$, $W_6 = 18$ and

$$(22) \qquad W_{3(a+2)} = 8\, W_{3(a+1)} - 14\, W_{3a} ,$$

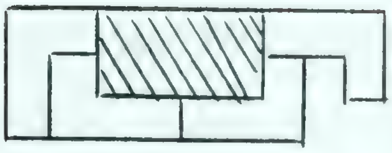
so that $\{W_{3n}\}$ has in part the form $\{4, 18, 88, 452, \ldots\}$.

Using P-pentominoes (a P-pentomino is a 2×3 rectangle with a unit square in one corner removed) the number of ways of packing a $5 \times n$ rectangle is given by

$$(23) \qquad W(x) = \frac{2x^2 + 2x^4 + 4x^6}{1 - 2x^2 - 2x^4 - 5x^6} = \sum_{n=1}^{\infty} W_n x^n ;$$

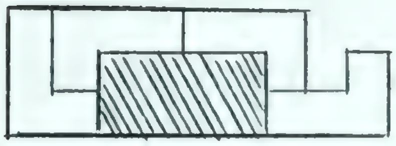
here $W_i = 0$ if $i = 2a$ and $\{W_{2n}\}$ has in part the form $\{2, 4, 16, 50, 152, 434, \ldots\}$.

While studying these problems, the following questio arose which we have been unable to settle. If f is a non-rectangular n-omino and f can be used to form $n \times \ell$ f-configurations, is it true that ℓ is always even ?

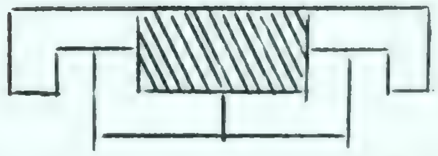


A

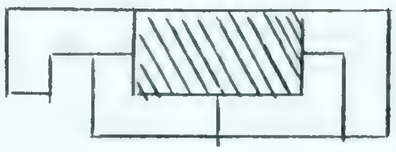
C



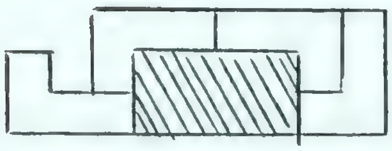
B



D



E



F

Figure 5. Every $3 \times 8k$ L-tetromino configuration is composed of a sequence of "links" labeled A, B, C, ... above. Such sequences are of the form {A, C, D, C, D, ..., C, E}, {A, C, D, C, D, ... , D, F}, {B, D, C, D, C, ... , C, E} or {B, D, C, D, C, ..., D, F} .

CHAPTER III

GENERATORS FOR SEQUENCES SATISFYING LINEAR DIFFERENCE EQUATIONS

Throughout this chapter we will be concerned with investigating the properties of a certain class C of sequences having terms generated by rational functions. (Here a function f(x) is said to generate a sequence $\{a_n\}$ if a_n is the coefficient of x^n in the Maclaurin series for f(x).) To be specific, a typical generator of a sequence in C is to consist of the quotient of two integral polynomials p(x) and q(x) which have no linear factor in common; if the degree of q(x) is k, we say the corresponding sequence is of the k^{th} order. From this definition of C it is easy to see that we are dealing with a set of sequences having integer terms which satisfy linear difference equations with constant coefficients (which we shorten to "linear difference equation" and mean the same thing).

We define addition and multiplication on the elements of C as term by term addition and multiplication of the sequences; thus, if $\{a_n\}$ and $\{b_n\}$ are in C we have

(1) $$\{a_n\} + \{b_n\} = \{a_n + b_n\} ,$$

(2) $$\{a_n\} \cdot \{b_n\} = \{a_n \cdot b_n\} .$$

If we include in C the sequence $\{0\}$ having all of its terms equal to zero, we can prove

Theorem 1: The set C is a commutative ring under the operations defined in (1) and (2).

Proof: The only aspect of the proof which is not at once obvious is that C is closed under multiplication; to prove this closure property

suppose

$$(3) \qquad f(x) = \sum_{m=0}^{\infty} a_n x^n$$

is convergent for $|x| < R$ and

$$(4) \qquad g(x) = \sum_{m=0}^{\infty} b_n x^n$$

is convergent for $|x| < R'$. We are interested in the function represented by the power series

$$(5) \qquad h(x) \quad \sum_{m=0}^{\infty} a_n \cdot b_n x^n$$

which by the Cauchy-Hadamard root test converges for $|x| < RR'$ and perhaps for a larger radius. Following Titchmarsh [16] we will obtain

$$(6) \qquad h(x) = \frac{1}{2\pi i} \int_\gamma f(s)\, g(\frac{x}{s})\, \frac{ds}{s} \quad ,$$

where γ is a contour, including the origin. on which $|s| < R$, $|\frac{x}{s}| < R'$. To prove (6) we substitute the power series for $g(\frac{x}{s})$ given by (4) into the integral in (6) and recalling that the series is uniformly convergent we integrate term by term; thus,

$$(7) \qquad \frac{1}{2\pi i} \int_\gamma f(s)\, g(\frac{x}{s})\, ds/s =$$

$$\frac{1}{2\pi i} \int_\gamma \left\{ f(s) \sum_{m=0}^{\infty} b_n\, x^n / s^{m+1} \right\} ds =$$

$$\sum_{m=0}^{\infty} a_n \cdot b_n x^n = h(x) .$$

In the penultimate equality we have used the fact that the residue of

$$(8) \qquad f(s)/ s^{n+1} = \sum_{r=0}^{\infty} a_r s^{r-n-1}$$

is a_n and hence the value of the corresponding integral is $2\pi i\, a_n$.

Suppose now that f(x) and g(x) in (3) and (4) are rational functions and that $\phi_1, \ldots, \phi_i$, and $\theta_1, \theta_2, \ldots, \theta_k$ are the singularities of f and g respectively; that is, the points ϕ_i and θ_i are the zeros of the polynomials in the denominators of f and g. Now we must have $|\phi| \geq k > |s|$ and $|\theta_i| \geq R' > |\frac{x}{s}|$ on γ; i.e., the points ϕ_i lie outside γ and x/θ_i lie inside γ (since $|s| > |x/\theta_i|$).

Using the fact that f and g are rational functions we can write the integral in (7) as a sum of integrals of partial fractions; thus,

$$(9) \qquad h(x) = \frac{1}{2\pi i} \int_\gamma f(s)g(x/s)ds/s$$

$$= \sum_{m=1}^{j} \frac{1}{2\pi i} \int_\gamma \frac{A_n(x)}{s - \theta_n} ds + \sum_{m=1}^{k} \frac{1}{2\pi i} \int_\gamma \frac{B_n(x)}{s - x/\theta_n} ds$$

But since each of the points θ_n is outside γ, each integral in the first sum is zero by Cauchy's Theorem, and since x/θ_n is inside γ, the value of each integral in the last sum is $2\pi i B_n(x)$. We have therefore

$$(10) \qquad h(x) = \sum_{m=1}^{k} B_n(x) \quad .$$

where each $B_n(x)$ is a rational function determined in the process of expressing $f(s)g(x/s)s$ as a sum of linear partial fractions. It is easy to see that each of the rational functions $B_n(x)$ generates a sequence in C so that the sum h(x) is also in C by the previous argument.

Having shown that C forms a ring, we will now turn attention to finding generators of sequences which are products (in the sense of (2)) of other sequences in C. Perhaps the most interesting case to be considered is when a sequence is a power of another sequence. Riordan [14] considered this problem for Fibonacci numbers, and Carlitz [2] extended his results to arbitrary sequences of the second order. It is instructive to study these methods before proceeding to the problem for sequences of order higher than the second. Our results and methods duplicate much of what is to be found in the paper of Carlitz noted above; however, the presentation and a few of the formulae are new. Results concerning sequences of order higher than the second seem to be new also.

Suppose the sequence $\{h_n\}$ has the generating function

$$h(x) = \{h_o + (h_1 - \alpha h_o)x\}\{1 - \alpha x - \beta x^2\}^{-1}, \tag{11}$$

so that the terms of the sequence $h_o, h_1, \ldots$ satisfy the linear difference equation

$$h_{n+2} = \alpha h_{n+1} + \beta h_n \qquad (n = 0, 1, \ldots). \tag{12}$$

We note in passing that the elements of $\{h_n\}$ and α and β may be taken from any ring; in particular, for C we have the integer ring in mind. We differ from Carlitz at this point in that no special values are assigned to h_o and h_1; the results are more elegant but less general if we put $h_o = 1$, $h_1 = \alpha$, as Carlitz does.

Letting θ_1 and θ_2 denote the roots of the equation $y^2 - \alpha y - \beta = 0$ associated with the polynomial in the denominator

of (12) (we assume $\theta_1 \neq \theta_2$ throughout), so that $\theta_1 + \theta_2 = \alpha$, $\theta_1\theta_2 = -\beta$. Now $h(x)$ can be split into two linear partial fractions and these in turn can be expanded into geometric series which when added together term by term give

$$(13) \qquad h_n = A_1\theta_1^{\ n} + A_2\theta_2^{\ n} \qquad (n = 0, 1, \ldots)$$

the explicit values of A_1 and A_2 are easily determined as the numerators of the two linear partial fractions whose sum is $h(x)$, but we will not need them in what follows.

From (13) we find

$$(14) \qquad h_n^k = \sum_{r=0}^{k} \binom{k}{r} A_1^{k-r} A_2^r (\theta_1^{k-r} \theta_2^r)^n \qquad (n = 0, 1, \ldots)$$

and from this expression it becomes obvious that $\{h_n^k\}$ has a generating function

$$(15) \qquad P_k(x) / Q_k(x) ,$$

where $P_k(x)$ is a polynomial with integer coefficients of degree less than or equal to k and

$$(16) \qquad Q_k(x) = \prod_{r=0}^{k} (1 - \theta_1^{k-r} \theta_2^r x) .$$

Since $Q_k(x)$ is a symetric polynomial in θ_1 and θ_2, it has coefficients which may be expressed as integral polynomials in α and β which we will denote by $A_k(r)$ in

$$(17) \qquad Q_k(x) = \sum_{r=0}^{k+1} A_k(r)x^r .$$

Pairing conjugate factors in the product in (16) gives $Q_k(x)$ as a product of integral quadratic forms when k is odd and includes a linear form if k is even. Thus,

$$(18) \qquad Q_k(x) = [(1-\theta_1^k x)(1-\theta_2^k x)][(1-\theta_1^{k-1}\theta_2 x)(1-\theta_1\theta_2^{k-1}x)]\ldots$$

$$= [1-(\theta_1^k+\theta_2^k)x + \theta_1^k\theta_2^k x^2][1-\theta_1\theta_2(\theta_1^{k-2}+\theta_2^{k-2})x + \theta_1^k\theta_2^k x^2]\ldots$$

But $\{\theta_1^n + \theta_2^n\} = \{H_n\}$ is a sequence which may be defined

$$(19) \qquad H_o = 2,\ H_1 = \alpha,\ H_{n+2} = \alpha H_{n+1} + \beta H_n \quad (n = 0, 1, \ldots),$$

so that writing H_n for $\theta_1^n + \theta_2^n$ and $-\beta$ for $\theta_1\theta_2$ in (18) one obtains (20) and (21) for odd and even k respectively.

$$(20) \qquad Q_{2s+1}(x) = \prod_{r=0}^{s} (1-(-\beta)^r H_{2(s-r)+1}x - \beta^{2s+1}x^2)$$

$$(21) \qquad Q_{2s}(x) = (1+(-\beta)^s x) \prod_{r=0}^{s-1} (1-(-\beta)^r H_{2(A-r)}x + \beta^{2s}x^2$$

Using (20) and (21) it is easy to verify the identity

$$(22) \qquad Q_k(x) = (1 - H_k x + (-\beta)^k x^2)\, Q_{k-2}(-\beta x)\ ;$$

substituting expressions for $Q_k(x)$ and $Q_{k-2}(-\beta x)$ given by (17) and performing the multiplication in the right member gives

$$(23) \qquad \sum_{r=0}^{b+1} A_k(r)x^r = A_{k-2}(0) - [\beta A_{k-2}(1) + H_k A_{k-2}(0)]x$$

$$+ \sum_{r=2}^{k-1} [(-\beta)^r A_{k-2}(r) - (-\beta)^{r-1} H_k A_{k-2}(r-1)+(-\beta)^{k+r-2}A_{k-2}(r-2)]x^r$$

$$+ [(-\beta)^{2k-2}A_{k-2}(k-2) - (-\beta)^{k-1}H_k A_{k-2}(k-1)]x^k$$

$$- (\beta)^{2k-1} A_{k-2}(k-1)x^{k+1}.$$

Equation (16) can be used to find $Q_1(x)$ and $Q_2(x)$ as follows:

$$(24) \qquad Q_1(x) = 1 - \alpha x - \beta x^2$$

$$(25) \qquad Q_2(x) = (1 - \theta_1^2 x)(1 - \theta_2^2 x)(1 - \theta_1\theta_2 x)$$

$$= 1 - (\alpha^2 + \beta)x - \beta(\alpha^2 + \beta)x^2 + \beta^2 x^3$$

Now using (24) and (25) we have by definition

$$(26) \qquad A_1(0) = 1, \quad A_1(1) = -\alpha, \quad A_1(2) = -\beta,$$

$$(27) \qquad A_2(0) = 1, \quad A_2(1) = -(\alpha^2+\beta), \; A_2(2) = -\beta(\alpha^2+\beta), \; A_2(3) = \beta^3.$$

Equating coefficients of like powers of α in (23) gives different equations for $A_k(r)$; namely,

$$(28) \qquad A_k(0) = A_{k-2}(0),$$

$$(29) \qquad A_k(1) = -[\beta A_{k-2}(1) + H_k A_{k-2}(0)],$$

$$(30) \qquad A_k(k+1) = (-\beta)^{2k-1} A_{k-2}(k-1),$$

$$(31) \qquad A_k(k) = \beta^{2k-2} A_{k-2}(k-2) - (-\beta)^{k-1} H_k A_{k-2}(k-1),$$

$$(32) \qquad A_k(r) = (-\beta)^r A_{k-2}(r) - (-\beta)^{r-1} H_k A_{k-2}(r-1)$$

$$+ (-\beta)^{k+r-2} A_{k-2}(r-2) \qquad \text{for } r = 2,3,\ldots,k-1.$$

The information contained in the relations (26) through (32) can be used to construct $Q_k(x)$ for any given k. The integral polynomials in α and β which are the A_k's are in themselves rather interesting; many relationships exist between them and the H_k's which can be proved by the latter formulae. Perhaps some of these relationships

will be evident in the table below in which one finds the coefficients of $Q_k(x)$ for $k = 1, 2, 3,$ and 4 in an array having the form

$$\begin{array}{lc} & A_1(0)\ \ A_1(1)\ \ A_1(2) \\ (33) & A_2(0)\ \ A_2(1)\ \ A_2(2)\ \ A_2(3) \\ & A_3(0)\ \ \ldots \end{array}$$

TABLE I: COEFFICIENTS OF $Q_k(x)$

$$1 \quad -\alpha \quad -\beta$$

$$1 \quad -(\alpha^2+\beta) \quad -\beta(\alpha^2+\beta) \quad \beta^3$$

$$1 \quad -(\alpha^4+3\alpha^2\beta+\beta^2) \quad -\beta(\alpha^6+5\alpha^4\beta+7\alpha^2\beta^2+2\beta^3) \quad \ldots$$

$$\ldots \quad \beta^3(\alpha^6+5\alpha^4\beta+7\alpha^2\beta^2+2\beta^3) \quad \beta^6(\alpha^4+3\alpha^2\beta+\beta^2) \quad -\beta^{10}$$

When $\alpha = \beta = 1$, the difference (12) is satisfied by Fibonacci or Lucas sequences; in this case the coefficients of $Q_k(x)$ become those given in

TABLE II

$$1 \quad -1 \quad -1$$

$$1 \quad -2 \quad -2 \quad 1$$

$$1 \quad -3 \quad 6 \quad 3 \quad 1$$

$$1 \quad -5 \quad -15 \quad 15 \quad 5 \quad -1$$

$$1 \quad -8 \quad -40 \quad 60 \quad 40 \quad -8 \quad -1$$

$$\ldots$$

By means of formulae (26) - (32) it is easy to prove the following theorems involving the numbers $A_k(r)$.

<u>Theorem 2</u>: $A_k(r)\beta^K = A_k(k - r + 1)$, where $K = [(k+1)k - 2r]/2$.

This theorem notes the symmetry evident in Tables I and II.

<u>Theorem 3</u>: $A_k(0) = 1$, $A_k(k+1) = (-1)^{(k+1)(k+2)/2}\ \beta^{k(k+1)/2}$

<u>Theorem 4</u>: $A_k(1) = \alpha A_{k-1}(1) + \beta A_{k-2}(1)$; that is, $\{A_k(1)\}$ is a sequence which satisfies (12).

So far we have indicated in some detail what can be learned about generating functions associated with powers of second order sequences; however, these methods also apply to sequences of order higher than the second, and in the remaining part of this chapter we will indicate some of this generality.

Using (9) and (10) we can find

$$\text{(34)} \qquad h_{k+1}(x) = \frac{1}{2\pi i} \int_\gamma \frac{[sh_o + (h_1 - \alpha h_o)x]\, h_k(s)}{s^2 - \alpha sx - \beta x^2}$$

$$= \frac{A_1}{2\pi i} \int_\gamma \frac{ds}{s - \theta_1 x} + \frac{A_2}{2\pi i} \int_\gamma \frac{ds}{s - \theta_2 x}$$

$$+ \sum_{r=0}^{k+1} \frac{B_r}{2\pi i} \int_\gamma \frac{ds}{s - \phi_r}$$

$$= A_1 + A_2 ,$$

where ϕ_r ($r = 0, 1, \ldots, k+1$) denotes the singularities of $h_k(s)$. But since $\theta_1 \neq \theta_2$ we can write

(35) $$A_1 = \lim_{s \to \theta_1 x} [sh_o + (h_1 - \alpha h_o)x]\, h_k(s)/(s - \theta_2 x)$$

$$= [(\theta_1 - \alpha)h_o + h_1]\, h_k(\theta_1 x)/(\theta_1 - \theta_2)$$

(36) $$A_2 = \lim_{s \to \theta_2 x} [sh_o + (h_1 - \alpha h_o)x] h_k(s)/(s - \theta_1 x)$$

$$= [(\theta_2 - \alpha)h_o + h_1] h_k(\theta_2 x)/(\theta_2 - \theta_1)$$

Substituting the final expressions for A_1 and A_2 given in (35) and (36) and writing $\theta_1 + \theta_2$ for α, we obtain the functional equation for $\{h_k(x)\}$

(37) $$h_{k+1}(x) = \frac{h_1 - \theta_2 h_o)h_k(\theta_1 x) - (h_1 - \theta_1 h_o)h_k(\theta_2 x)}{\theta_1 - \theta_2}$$

If $h_o = 1$ and $h_1 = \alpha$, (37) becomes

(38) $$h_{k+1}(x) = [\theta_1 h_k(\theta_1 x) - \theta_2 h_k(\theta_2 x)]/(\theta_1 - \theta_2) .$$

We observe that $h_o(x) = (1 - x)^{-1}$ for all sequences, so that in particular when $k = -1$ in (37) or (38) the expressions become

(39) $$(\theta_1 - \theta_2)(1 - x)^{-1} = (h_1 - \theta_2 h_o)h_{-1}(\theta_1 x) - (h_1 - \theta_1 h_o)h_{-1}(\theta_2 x),$$

(40) $$(\theta_1 - \theta_2)(1 - x)^{-1} = \theta_1 h_{-1}(\theta_1 x) - \theta_2 h_{-1}(\theta_2 x),$$

where of course

(41) $$h_{-1}(x) = \sum_{m=0}^{\infty} x^n/h_n .$$

Substituting the power series in (41) for $f_{-1}(\theta_1 x)$ and $f_{-1}(\theta_2 x)$, we can verify (39) and (40) directly. In fact, (37) can be verified in this way for any integer k. This observation has raised a question which goes beyond the scope of the present work; to wit, does there exist a formula in closed form for

$$(42) \qquad H_\theta(x) = \sum_{n=0}^{\infty} h_n^\theta x^n ,$$

where θ is any complex number, and in particular a negative integer? When θ is a non-negative integer we have proved h_θ to be a rational function; I have been able to make only slight progress in the case when θ is chosen otherwise.

Using (38), (the same can be done for the more general expression in (37), but the result is less elegant) one obtains

$$(43) \qquad h_{k+1}(x) = (\theta_1-\theta_2)^{-2} \sum_{r=0}^{2} (-1)^r \theta_1^{2-r} \theta_2^r h_{k-1}(\theta_1^{2-r}\theta_2^r)$$

by substituting the expressions for $h_k(\theta_1 x)$ and $h_k(\theta_2 x)$ given by (38) into the same relation. Iterating this process (and writing k for k+1) gives

$$(44) \qquad h_k(x) = (\theta_1-\theta_2)^{-t} \sum_{r=0}^{t} (-1)^r \binom{t}{r} \theta_1^{t-r}\theta_2^r h_{k-t}(\theta_1^{t-r}\theta_2^r x)$$

which is easily verified by induction on t. In particular, since $h_o(x) = (1-x)^{-1}$ we can take $k = t$ in (44) to obtain an explicit formula for $h_k(x)$; namely,

$$(45) \qquad h_k(x) = (\theta_1-\theta_2)^{-k} \sum_{r=0}^{k} (-1)^r \binom{k}{r} \theta_1^{k-r}\theta_2^r (1 - \theta_1^{k-r} \theta_2^r x)^{-1} .$$

From earlier results we know $h_k(x)$ is an element of C, but this is also evident in (45) if one sums the linear partial fractions involved. One can also obtain the explicit formula for $P_k(x)$ in $h_k(x) = P_k(x)/Q_k(x)$.

There is yet another method for finding the generating functions of the product of two sequences in C; here we will further exploit an idea of Gould's [10]. Suppose $A(1 - \theta x)^{-1}$ generates $\{A_n\} = \{A\theta^n\}$ and $f(x)$ generates $\{b_n\}$; since $\{A_n . b_n\} = \{A\theta^n b_n\}$, it is clear that a $f(\theta x)$ generates the product of the sequences in this case. Now suppose $\{A_n\}$ is generated by

$$(46) \qquad a(x) = \sum_{r=1}^{k} A_r (1 - \theta_r x)^{-1} \quad ,$$

so that

$$(47) \qquad \{a_n . b_n\} = \{\sum_{r=1}^{k} A_r \theta_r^n . b_n\}$$

$$= \sum_{r=1}^{k} \{A_r \theta_r^n . b_n\} \; ,$$

which by the previous argument has the generator

$$(48) \qquad c(x) = \sum_{r=1}^{k} A_r \, f(\theta_r x).$$

Here we have used the distributive law in C and the fact that

$$(49) \qquad a_n = \sum_{r=1}^{k} A_r \theta_r^r \; .$$

So far we have not shown that the generator of the product of any two sequences in C can be found by this method, since it was assumed that at least one of the generating functions involved had distinct singularities. To treat this problem we need only find the generator of $\{a_n . b_n\}$ where $f(x)$ generates $\{b_n\}$ and $a(1 - \theta x)^{-k}$ generates $\{a_n\}$, and proceed as in (46) - (48).

Since

$$(50) \qquad a(1 - \theta x)^{-k-1} = \sum_{n=0}^{\infty} a\theta^n \binom{n+k}{k} x^n ,$$

we see at once that

$$(51) \qquad \sum_{n=0}^{\infty} a\theta^n b_n \binom{n+k}{k} x^n = \frac{a}{k!} D_x^k \{x^k f(\theta x)\} ,$$

where D_x denotes the differential operator.

To demonstrate the usefulness of this method, we will find a recursion relation satisfied by the generators $f_k(x)$ of $\{f_n^k\}$ where

$$(52) \qquad f_1(x) = \sum_{r=1}^{t} A_r (1 - \theta_r x)^{n_r} .$$

Since $\{f_n^{k+1}\} = \{f_n . f_n^k\}$ and

$$(53) \qquad f_n = \sum_{r=1}^{t} A_r \theta_r^n \binom{n + n_r - 1}{n_r - 1} f_n^k ,$$

we have

$$(54) \qquad f_n^{k+1} = \sum_{r=1}^{t} A_r \theta_r^n \binom{n + n_r - 1}{n_r - 1} f_n^k ,$$

so that by using (51) we obtain

$$(55) \qquad f_{k+1}(x) = \sum_{r=1}^{t} \frac{A_r}{(n_r-1)!} D_x^{n_r-1} \{ x^{n_r-1} f_k(\theta_r x).$$

If we take $f_1(x)$ in (52) as the generator of the Fibonacci sequence, (55) simply becomes

$$(56) \qquad f_{k+1}(x) = [\theta_1 f_k(\theta_1 x) - \theta_2 f_k(\theta_2 x)]/(\theta_1 - \theta_2).$$

CHAPTER IV

DETERMINANTS WITH CERTAIN K^{th} POWER ELEMENTS

Introductory Remarks: In Chapter III, we defined a class C of sequences of integers which had generators consisting of the quotients of integral polynomials. We showed that C is a ring under term by term addition and multiplication of the sequences. Many of the observations we have made concerning C can be carried over to a class K of sequences having terms selected from an arbitrary integral domain D; the generators of these sequences are quotients of elements taken from D[y], the polynomial domain formed by adjoining the free variable y to D. It is easy to see that any class K defined in this way forms a ring under term by term addition and multiplication for the same reason that C forms a ring under these operations. In what follows, we will fix our attention on properties of sequences having elements in the integer domain Z, or in the polynomial domain Z[x] but it will be obvious that our remarks apply in the general case as well.

A Class of Determinants: Suppose $\{p_n\}$ is a second order sequence having elements in Z or Z[x]; that is,

$$(1) \qquad p_{n+2} = \alpha p_{n+1} + \beta p_n \qquad n = 0, 1, \ldots ,$$

and define

$$(2) \qquad P_k(n) = \begin{vmatrix} p_n^k & p_{n+1}^k & \cdots & p_{n+k}^k \\ p_{n+1}^k & p_{n+2}^k & \cdots & p_{n+k+1}^k \\ \vdots & \vdots & & \vdots \\ p_{n+k}^k & p_{n+k+1}^k & \cdots & p_{n+2+k}^k \end{vmatrix}$$

We know from our results in Chapter III that $\{p_n^k\}$ is a $(k + 1)^{st}$ order sequence since $\{p_n\}$ is a second order sequence; it is important to note at this point that the order of $\{p_n^k\}$ is also the order of $P_k(n)$. Now we have

$$(3) \qquad p_n^k = - \sum_{r=1}^{k+1} A_k(r)\, p_{n-r}^k \qquad n = k+1,\ k+2,\ \dots ,$$

where the A's are certain elements taken from the same domain as the p's.

Using (3) and certain standard operations on determinants, it is not difficult to alter $P_k(n)$ in order to obtain $P_k(n+1)$; in doing this, we can also note the change in the value of the determinants. We operate on $P_k(n)$ as follows: interchange the first column with the second, the second column with the third, and so on until the columns have been shifted one column to the left and the first column has become the last; k columns have been interchanged, so the value of the new determinant is $(-1)^k\, P_k(n)$. Multiplying the last column of this determinant by $-A_k(k+1)$ and adding appropriate multiples of the other columns to it (these multiples are the A's in (3)) we obtain the last column of $P_k(n+1)$, so that the altered determinant is $P_k(n+1)$ and has the value $(-1)^{k+1}\, A_k(k+1)\, P_k(n)$. A trivial induction argument on n gives

$$(4) \qquad P_k(n) = \{(-1)^{k+1} A_k(k+1)\}^{n-1}\, P_k(1) ;$$

using the value of $A_k(k+1)$ given by Theorem 3 of Chapter III, we obtain

<u>Theorem 1</u>:

$$(5) \qquad P_k(n) = \{(-1)^{(k+1)(k+4)/2}\, \beta^{k(k+1)/2}\}^{n-1}\, P_k(1).$$

The cases when $\beta = \pm 1$ in (5) are particularly elegant. When $\beta = -1$, the exponent of (-1) in (5) becomes $(k+1)(k+2)$ which is always even so that

<u>Corollary 1</u>: if $\beta = -1$, $P_k(n) = P_k(1)$.

When $\beta = 1$, the exponent of (-1) becomes $(n-1)(k+1)(k+4)/2$ so that

<u>Corollary 2</u>: if $\beta = 1$, $P_k(n) = (-1)^{(n-1)(k+1)(k+4)/2} P_k(1)$.

<u>An Example Involving a Sequence of Polynomials</u>: Theorem 1 gives a generalization of the work of several authors. For example, Lorch and Moser [11] proposed that one prove

$$(6) \qquad R_1(n) = \begin{vmatrix} r_n & r_{n+1} \\ r_{n+1} & r_{n+2} \end{vmatrix} = x \qquad n = 0, 1, 2, \ldots$$

where $r_o = 1$ and

$$(7) \qquad r_n = \sum_{r=0}^{n} \binom{n+r}{n-r} x^r \qquad n = 1, 2, \ldots$$

It is not difficult to show that $\{r_n\}$ satisfies the linear difference equation

$$(8) \qquad r_{n+2} = (x+2)\, r_{n+1} - r_n \qquad n = 0, 1, 2, \ldots$$

since $\beta = -1$, we can apply the first corollary to Theorem 1 and find that

$$(9) \qquad R_1(n) = R_1(0) = \begin{vmatrix} 1 & 1+x \\ 1+x & 1+3x+x^2 \end{vmatrix} = x .$$

We can go further than (9) however and show that

$$(10) \qquad R_2(n) = 2x^3(x + 2)^2 = R_2(0) ;$$

in fact, since $R_k(n) = R_k(0)$ for all non-negative integers k and n, our attention is drawn to finding $R_k(0)$. This task is made easier if we extend $\{r_n\}$ to include terms with negative rank so that the resulting sequence still satisfies (8) for all integers n. This is done by writing

$$(11) \qquad r_n = (x + 2)r_{n+1} - r_{n+2} ,$$

so that it becomes evident that $r_{-n} = r_{n-1}$. Since we have $R_k(i) = R_k(j)$ for all integers i and j, we can now select a convenient n (i.e., one which minimizes the "size" of the r^k's. This observation leads us to select n = -k which gives a nearly circulent determinant:

$$(12) \qquad \begin{vmatrix} r^k_{-k} & r^k_{-k+1} & \cdots & r^k_o \\ r^k_{-k+1} & r_{-k+2} & \cdots & r^k_1 \\ \vdots & \vdots & & \vdots \\ r^k_o & r^k_1 & \cdots & r^k_k \end{vmatrix} =$$

$$\begin{vmatrix} r^k_{k-1} & r^k_{k-2} & \cdots & r^k_o & r^k_o \\ r^k_{k-2} & r^k_{k-3} & \cdots & r^k_o & r^k_1 \\ \vdots & \vdots & & \vdots & \vdots \\ r^k_o & r^k_1 & \cdots & r^k_{k-1} & r^k_k \end{vmatrix}$$

Thus, when $k = 2$ we compute $R_2(-2)$ instead of $R_2(0)$; these determinants are respectively

$$(13) \qquad R_2(-2) = \begin{vmatrix} (x+1)^2 & 1 & 1 \\ 1 & 1 & (x+1)^2 \\ 1 & (x+1)^2 & (x^2+3x+1)^2 \end{vmatrix} = 2x^3(x+2)^2$$

$$(14) \qquad R_2(0) = \begin{vmatrix} 1 & (x+1)^2 & (x^2+3x+1)^2 \\ (x+1)^2 & (x^2+3x+1)^2 & (x^3+5x^2+6x+1)^2 \\ (x^2+3x+1)^2 & (x^3+5x^2+6x+1)^2 & (x^4+7x^3+15x^2+10x+1)^2 \end{vmatrix}$$

<u>An Example Involving the Fibonacci Sequence</u>: In Chapter III, we discussed $\{f_n^k\}$, the sequence of the k^{th} powers of the Fibonacci numbers. We define $F_k(n)$ so that it is analogous to $P_k(n)$ in (2). Since $f_{n+2} = f_{n+1} + f_n$, we have $\beta = 1$, and the second corollary of Theorem 3 applies.

When $k = 1$, we obtain a well known result:

$$(15) \qquad F_1(n) = \begin{vmatrix} f_n & f_{n+1} \\ f_{n+1} & f_{n+2} \end{vmatrix} = (-1)^{5(n-1)} = (-1)^{n-1}$$

When $k = 2$, we obtain a result noted in [1]:

$$(16) \qquad F_2(n) = \begin{vmatrix} f_n^2 & f_{n+1}^2 & f_{n+2}^2 \\ f_{n+1}^2 & f_{n+2}^2 & f_{n+3}^2 \\ f_{n+2}^2 & f_{n+3}^2 & f_{n+4}^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 4 \\ 1 & 4 & 9 \\ 4 & 9 & 25 \end{vmatrix} \cdot (-1)^{9(n-1)} = 2(-1)^{n-1} .$$

Using the same idea used to obtain (13), we can compute $F_3(n) = 36$, or any of various other $F_k(n)$ for small k's. The value of $F_k(0)$ for arbitrary k remains an open question.

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